Guaranteed plate eigenvalues with adaptivity

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including joint work with Carsten Carstensen (HU Berlin)

Shapes and eigenvalues

- What you hear is a superposition of the drum's eigenfrequencies ω

$$-\Delta u = \omega^2 u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \qquad\qquad (\Delta = \partial_{xx}^2 + \partial_{yy}^2)$$

- Weyl's law: eigenvalues encode area $|\Omega|$ and circumference $|\partial\Omega|$

$\Rightarrow$ 'Can one hear the shape of a drum?' [Kac 1966] Answer: NO!

Outline

Plate eigenvalues

Variational eigenvalue problem

Guaranteed upper eigenvalue bounds

Guaranteed lower eigenvalue bounds

Plate eigenvalues

Can one hear the shape of a plate?

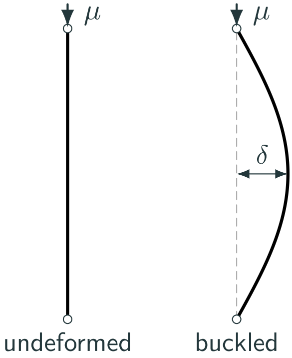
Plates: *thin elastic bodies* such as metal/glass sheets, airplane fuselage, concrete floors...

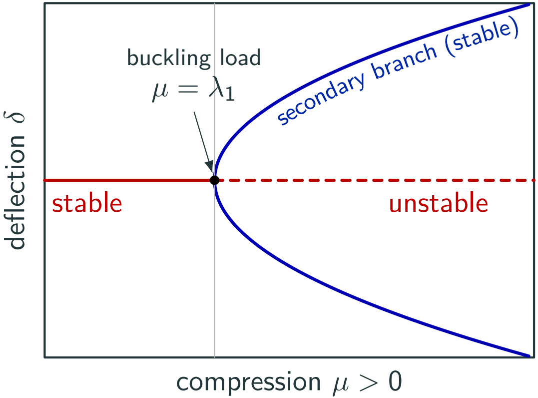
Chladni figures cropped from "Prato de Chladni 02" by Matemateca (IME USP) / R. T. Argenton, CC BY-SA 4.0

- vibration governed by bending moments described by biharmonic eigenvalues

$$\Delta^2 u = \lambda u \qquad \text{in } \Omega \qquad + \textit{boundary conditions}$$

Buckling of plates





buckling occurs after load surpasses the critical buckling eigenvalue $\mu = \lambda_1$

$$\Delta^2 u = \lambda B u \qquad \text{in } \Omega \qquad + \text{boundary conditions}$$

Eigenvalue computation matters in practice...

Safety-critical engineering applications require certified bounds to obtain admissible operating regimes and structural safety margins, e.g.,

- natural vibration and resonance frequencies
- structural stability and safety analysis

$\Longrightarrow$ This talk: guaranteed and efficient plate eigenvalue computation

source: futurism.com

source: structuralguide.com

Disclaimer: We will *not* discuss relevant aspects of numerical linear algebra on certified computations! In particular, we ignore round-off errors and assume an exact solver for the discrete problem

Variational eigenvalue problem

biharmonic eigenvalue problem

bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ with polygonal boundary $\partial\Omega$

$$\begin{aligned} \Delta^2 u &= \lambda u && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_{\mathrm{S}} \subset \partial\Omega, && \text{(simply-supported)} \\ u = \partial_\nu u &= 0 && \text{on } \Gamma_{\mathrm{C}} \subset \partial\Omega && \text{(clamped)} \end{aligned}$$

classical interpretation (pointwise derivatives of $u \in C^4(\Omega)$) too strong and *not physical*

Integration by parts of $\Delta^2 u = \lambda u$ (Gauß divergence theorem)

Multiply PDE by test function $\varphi \in C_0^\infty(\Omega)$ and integrate by parts

$$\lambda \int_\Omega u\,\varphi\,\mathrm{d}x = \int_\Omega \underbrace{\Delta(\Delta u)}_{\operatorname{div}(\nabla \Delta u)}\,\varphi\,\mathrm{d}x = -\int_\Omega \nabla\Delta u \cdot \nabla\varphi\,\mathrm{d}x + \cancelto{0}{\int_{\partial\Omega} \partial_\nu \Delta u\,\varphi\,\mathrm{d}s}$$

$$= +\int_\Omega \Delta u\,\Delta\varphi\,\mathrm{d}x + \cancelto{0}{\int_{\partial\Omega} \Delta u\,\partial_\nu\varphi\,\mathrm{d}s}$$

$\Longrightarrow$Euler–Lagrange equations for the energy $\mathcal{E}(u) = \int_\Omega (\Delta u)^2\,\mathrm{d}x$ subject to $\int_\Omega u^2\,\mathrm{d}x = 1$ for

$u \in H^2(\Omega) := \{v \in L^2(\Omega) \ : \ D^2 v \in L^2(\Omega)\}$ ($\widehat{=}$ Hilbert space version of $C^2(\overline{\Omega})$)

Variational eigenvalue problem (EVP)

Boundary conditions prescibed in ansatz/trial space $V \subset H^2(\Omega)$

Seek eigenpair $(\lambda, u) \in \mathbb{R} \times V$ with normalised eigenfunction $\|u\|_{L^2(\Omega)} = 1$ and

$$a(u, v) = \lambda \, b(u, v) \qquad \text{for all } v \in V$$

with *symmetric, positive definite* bilinear forms $a(\,\bullet\,,\bullet\,),\; b(\,\bullet\,,\bullet\,) : V \times V \to \mathbb{R}$

$$\text{where} \quad a(v, w) := \int_\Omega \Delta v \, \Delta w \,\mathrm{d}x \qquad \text{and} \qquad b(v, w) := \int_\Omega v \, w \,\mathrm{d}x \qquad (v, w \in V)$$

Spectral properties and min–max principle

Symmetric EVP: all eigenvalues are real (and of finite multiplicity)

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_\ell \to \infty \qquad \text{as } \ell \to \infty,$$

corresponding eigenfunctions $(u_\ell)_{\ell=1}^{\infty} \subset V$ form an $L^2(\Omega)$-orthonormal basis for V

Main tool: Min–max/Rayleigh–Ritz principle

$$\lambda_m = \min_{\substack{W \subset V \\ \dim W = m}} \max_{0 \neq v \in W} \frac{a(v,v)}{b(v,v)}$$

Guaranteed upper eigenvalue bounds

Conforming EVP computes guaranteed upper bounds

Given $V_h \subset V$ finite-dimensional, seek $(\lambda_h, u_h) \in \mathbb{R} \times V_h$ with $\|u_h\|_{L^2(\Omega)} = 1$:

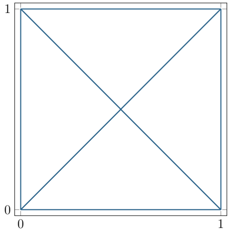
$$a(u_h, v_h) = \lambda_h \, b(u_h, v_h) \qquad \text{for all } v_h \in V_h \subset V$$

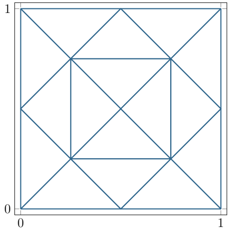
$N = \dim V_h$ discrete eigenvalues: $\quad 0 < \lambda_{h,1} \le \lambda_{h,2} \le \lambda_{h,3} \le \cdots \le \lambda_{h,N}$

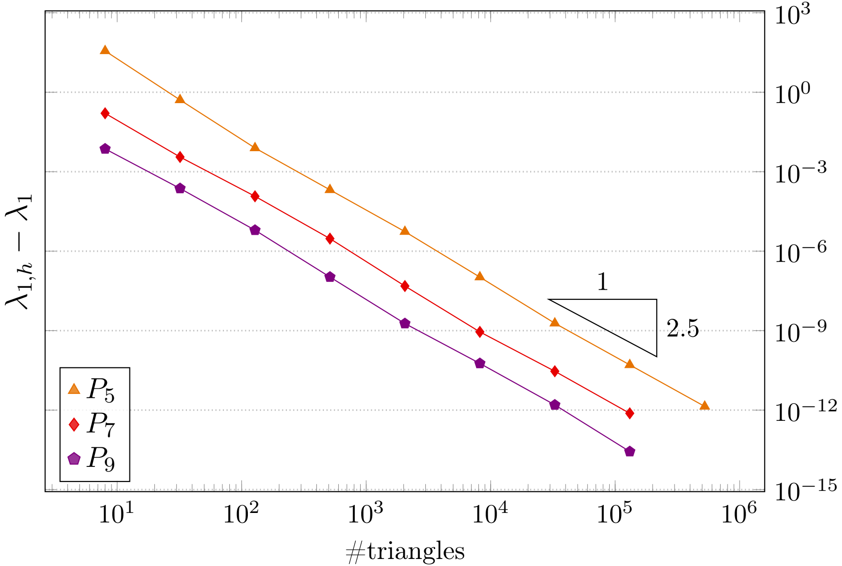
Min-max principle:

$$\boxed{\lambda_m = \min_{\substack{W \subset V \\ \dim W = m}} \max_{0 \neq v \in W} \frac{a(v,v)}{b(v,v)} \le \min_{\substack{W_h \subset V_h \\ \dim W_h = m}} \max_{0 \neq v_h \in W_h} \frac{a(v_h,v_h)}{b(v_h,v_h)} = \lambda_{h,m}}$$

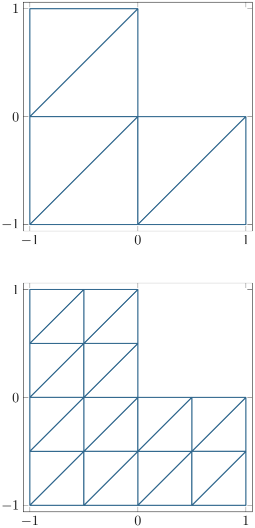
Experiment on the clamped unit square

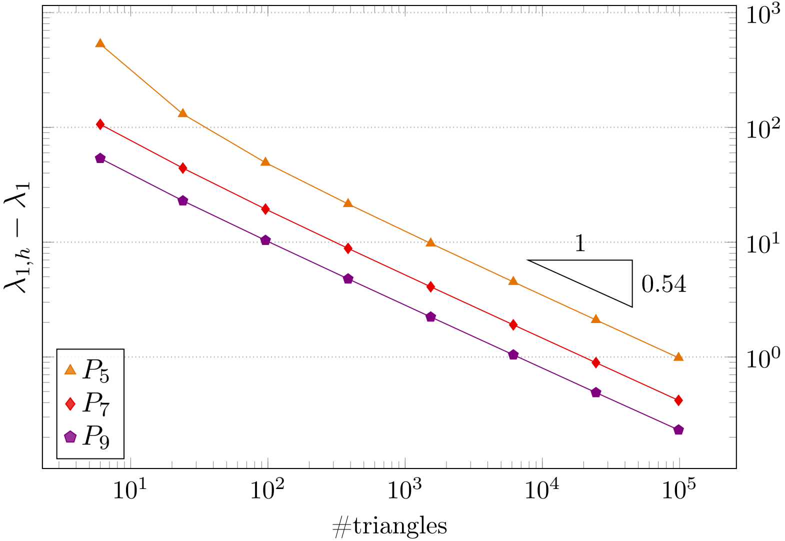






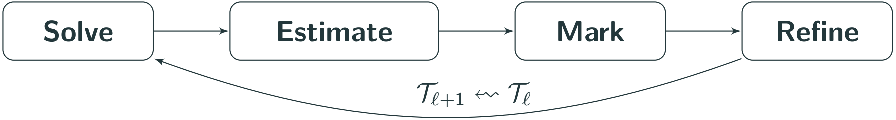
Slow convergence on L-shaped domain





Adaptive algorithm and optimal rates for simple eigenpair (λ, u)

AFEM:


Solve

Estimate

Mark

Refine

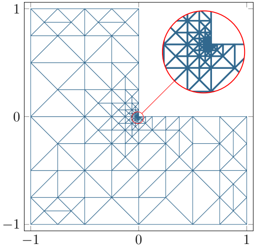
$\mathcal{T}_{\ell+1} \leftsquigarrow \mathcal{T}_\ell$

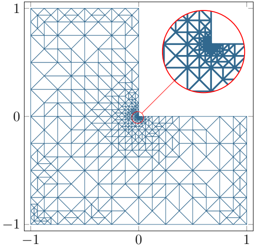
error estimator $$\eta^2(T) := |T|^2 \|\lambda_h u_h - \Delta^2 u_h\|^2_{L^2(T)} + |T|^{1/2} \sum_{E \in \mathcal{E}(T)} J_E(u_h) \qquad (T \in \mathcal{T})$$

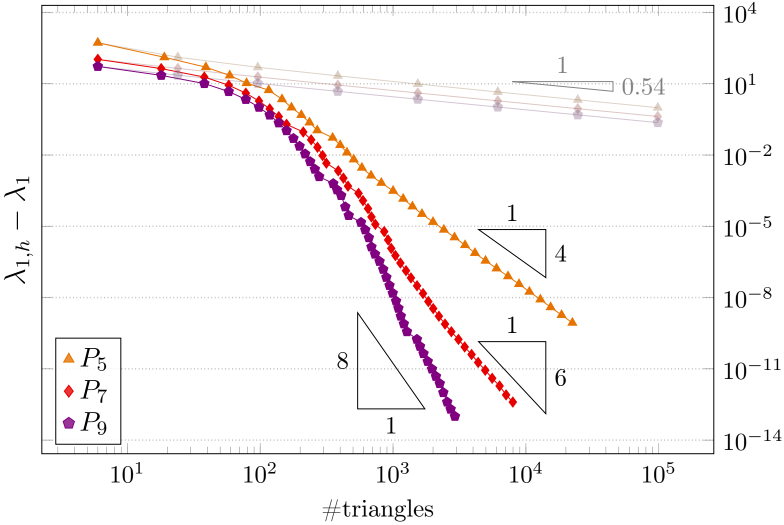
Theorem (Quasi-optimal convergence [Carstensen–BG (2025)])

Under usual assumptions (e.g., $h_{\max} \ll 1$ sufficiently small), the adaptive algorithm AFEM converges with the ***optimal rate*** *in the sense that any other marking strategy leads at most to the same convergence rate.*

AFEM recovers optimal rates

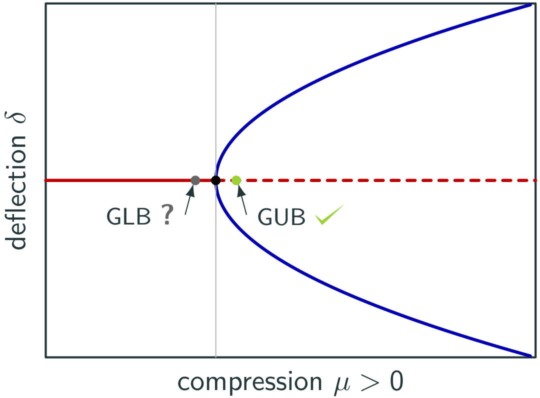






Guaranteed lower eigenvalue bounds

Back to buckling

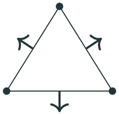


$V_h \subset V$ implies $\lambda \leq \lambda_h$: upper bounds ✓

lower bounds $\lambda_h \leq \lambda$?: break conformity!

Nonconforming plate element

$$V_h := \left\{ v_h \in P_2(\mathcal{T}) \,\middle|\, \begin{array}{rl} v_h & \text{continuous at vertices} + \text{BC}, \\ \partial_{\nu_E} v_h & \text{continuous at edge midpoints} + \text{BC} \end{array} \right\}$$



Natural extension $a_{\mathrm{pw}}(\bullet,\bullet) : (V + V_{\mathrm{nc}}) \times (V + V_{\mathrm{nc}}) \to \mathbb{R}$ of energy form $a = a_{\mathrm{pw}}|_{V \times V}$

Seek $(\lambda_h, u_h) \in \mathbb{R} \times V_h$ with $\|u_h\|_{L^2(\Omega)} = 1$:

$$a_{\mathrm{pw}}(u_h, v_h) = \lambda_h\, b(u_h, v_h) \qquad \text{for all } v_h \in V_h$$

$N = \dim V_h$ discrete eigenvalues: $0 < \lambda_{h,1} \le \lambda_{h,2} \le \lambda_{h,3} \le \cdots \le \lambda_{h,N}$

GLB from nonconforming FEM

Local interpolation operator $I : V \to V_h$ with

- orthogonality $((1-I)V \perp V_h)$: $\quad a_{\mathrm{pw}}((1-I)v, v_h) = 0 \qquad (v \in V, v_h \in V_h)$
- approximation property: $\quad \|(1-I)v\|_{L^2(\Omega)} \leq 0.074\, h_{\max}^2 \interleave (1-I)v \interleave_{\mathrm{pw}} \qquad (v \in V)$

with norms $\| \bullet \|_{L^2(\Omega)} = \sqrt{b(\bullet, \bullet)}$ and $\interleave \bullet \interleave_{\mathrm{pw}} = \sqrt{a_{\mathrm{pw}}(\bullet, \bullet)}$

Theorem (Carstensen et. al (2014+), Gallistl (2015), Liu et. al (2019))

$$GLB(\lambda_{h,m}) := \frac{\lambda_{h,m}}{1 + (0.074 h_{\max}^2)^2 \lambda_{h,m}} \leq \lambda_m \qquad (m \in \mathbb{N})$$

Proof for the principal eigenpair $(\lambda, u) = (\lambda_1, u_1)$

- $1 = \|u\|^2_{L^2(\Omega)} \leq (1+\delta^{-1})\|(1-I)u\|^2_{L^2(\Omega)} + (1+\delta)\|Iu\|^2_{L^2(\Omega)}$ $(\delta > 0)$
- $\lambda = a(u,u) = |||u|||^2_{\mathrm{pw}} = |||(1-I)u|||^2_{\mathrm{pw}} + |||Iu|||^2_{\mathrm{pw}}$
- $\lambda_h = \min\limits_{v_h \in V_h} \dfrac{a_{\mathrm{pw}}(v_h, v_h)}{b(v_h, v_h)} \leq \dfrac{a_{\mathrm{pw}}(Iu, Iu)}{b(Iu, Iu)} = \dfrac{|||Iu|||^2_{\mathrm{pw}}}{\|Iu\|^2_{L^2(\Omega)}} \quad \Rightarrow \quad \lambda_h \|Iu\|^2_{L^2(\Omega)} \leq |||Iu|||^2_{\mathrm{pw}}$

Hence $\lambda_h \leq (1+\delta^{-1})(0.074 h^2_{\max})^2 \lambda_h \, |||(1-I)u|||^2_{\mathrm{pw}} + (1+\delta)\, |||Iu|||^2_{\mathrm{pw}}$

This and $\delta = (0.074 h^2_{\max})^2 \lambda_h$ result in $\lambda_h \leq (1 + (0.074 h^2_{\max})^2 \lambda_h)\, \lambda$ $\square$

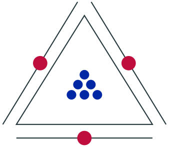
Limitations of nonconforming plate element

The nonconforming plate element is of lowest-order – no higher-order versions known

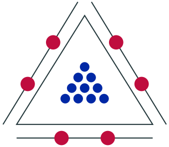
- $GLB(\lambda_h) := \frac{\lambda_h}{1+(0.074 h_{\max}^2)^2 \lambda_h}$ intrinsically low-order: $\lambda_h - GLB(\lambda_h) \in \mathcal{O}(h_{\max}^4)$
- quasi-optimal eigenvalue convergence $\lambda_h \to \lambda$ requires adaptive mesh-refinement vs. convergence of GLB requires $h_{\max} \to 0$

$\Rightarrow$ Higher-order GLB require modifications/another strategy!

Nonconforming idea generalised: $V_h = P_k(\mathcal{T}) \times P_{k-2}(\mathcal{E})^3$ $(k \geq 2)$



$k = 2 : \quad V_h = P_2(\mathcal{T}) \times P_0(\mathcal{E})^3$



$k = 3 : \quad V_h = P_3(\mathcal{T}) \times P_1(\mathcal{E})^3$

- L^2-orthogonal interpolation $\Pi = (\Pi_{\mathcal{T}}, \Pi_{\mathcal{E}}, \Pi_{\mathcal{E}}\nabla) : V \to V_h$ by L^2-orthogonal projection

$$b((1 - \Pi_{\mathcal{T}})v, p_k) = 0 \qquad (v \in V, p_k \in P_k(\mathcal{T}))$$

- weak Hessian: reconstruction operator $\mathcal{R} : V_h \to P_k(\mathcal{T})$ with

$$((D^2 - \mathcal{R} \circ \Pi)v, p_{k-2})_{L^2(\Omega)} = 0 \qquad (v \in V, p_{k-2} \in P_{k-2}(\mathcal{T})^{2\times 2})$$

GLB from skeletal schemes

Seek $(\lambda_h, u_h) \in \mathbb{R} \times V_h$ with $b_h(u_h, u_h) = 1$:

$$a_h(u_h, v_h) = \lambda_h\, b_h(u_h, v_h) \qquad \text{for all } v_h \in V_h$$

Theorem (Carstensen–BG–Zhai (2026))

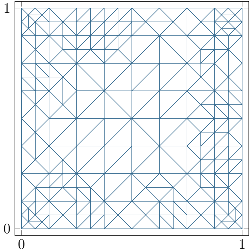
There exists a computable $\gamma_h(\lambda_h) \geq 0$ *with*

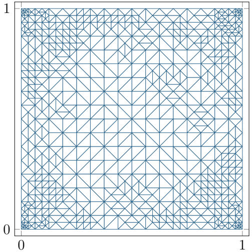
$$\frac{\lambda_h}{1 + \gamma_h(\lambda_h)} \leq \lambda.$$

For sufficiently small $h_{\max}$, *it holds* $\gamma_h(\lambda_h) = 0$. *Hence* $\lambda_h \leq \lambda$.

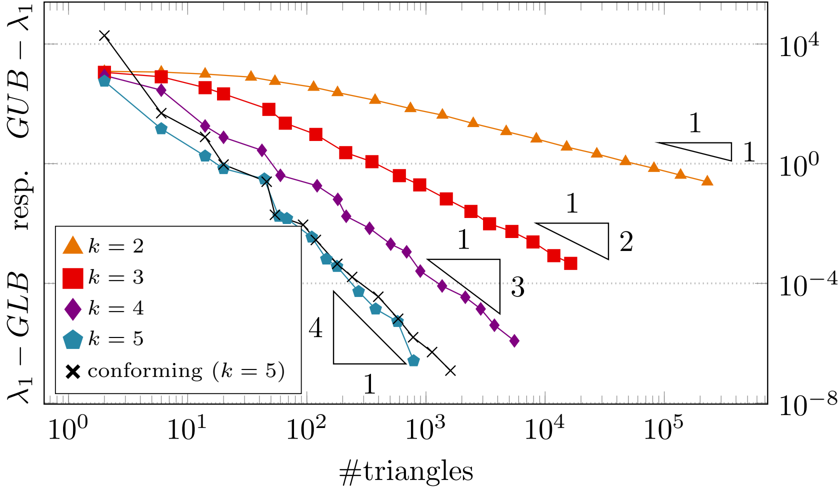
Similar method and GLB for skeletal schemes in Liang–Tran (2025, preprint)

Efficient guaranteed upper bounds (GUB) and lower bounds (GLB)

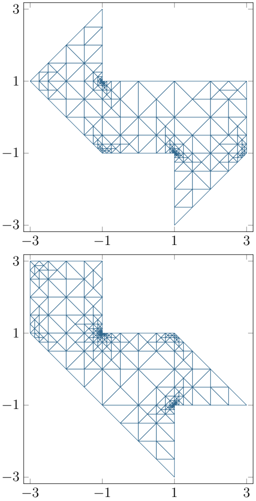




$k = 5$ and $|\mathcal{T}| > 200$: Eight guaranteed digits of $\lambda_1 = 1294.9339\ldots$



Can one hear the shape of a plate?



	simply-supported (both)	clamped (above)	clamped (below)
λ_1	10.36402107 . . .	28.05868638 . . .	25.01410502 . . .
λ_2	16.40218492 . . .	42.80755796 . . .	50.67098031 . . .
λ_3	36.28192141 . . .	73.51576421 . . .	72.08871248 . . .
λ_4	47.33781364 . . .	109.3553560 . . .	101.5078678 . . .
λ_5	61.23040947 . . .	123.8961882 . . .	119.5686100 . . .
λ_6	90.01373420 . . .	151.8305130 . . .	162.5851723 . . .
λ_7	119.0525060 . . .	227.0386261 . . .	217.9878528 . . .
λ_8	135.1118164 . . .	251.6305773 . . .	263.6684022 . . .
λ_9	152.2017047 . . .	290.2524354 . . .	297.5045897 . . .
λ_{10}	187.1231164 . . .	311.1681008 . . .	315.5017873 . . .

Numerical evidence ($>$ 20 digits): isospectrality for simply-supported BC

Numerical verification: both domains are *not* isospectral for clamped BC

Summary

- eigenvalues characterise, e.g., important structural properties of materials
- min–max/Rayleigh–Ritz principle as main tool for guaranteed bounds
  - *upper bounds*: conforming schemes $V_h \subset V$ — requires $V_h \subset C^1(\Omega)$
  - *lower bounds*: nonconforming/skeletal $V_h \not\subset V$ with built-in orthogonalities
- higher-order convergence requires adaptive mesh refinement (AFEM)
- numerical evidence for isospectral simply-supported plates—unknown for clamped plates

Open topics: nonlinear eigenvalue problems, quasi-optimal convergence for skeletal schemes, efficient (iterative/inexact) solvers, analysis of roundoff-errors (numerical linear algebra), ...

Thank you for your attention

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Berlin Mathematical School
BMS

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